Statistical properties of the solutions to the stochastic Green function

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Abstract.

The analysis of electromagnetic fields in chaotic closures has been studied with great interest in the last few years. The deterministic results in free space are well-known, so an attempt has been made to develop the stochastic Green function from them, using the Random Matrix Theory, the Berry's hypothesis and the works from Wigner about Gaussian distributions. This allows us, for example, to improve the WiFi signal inside a ship.

Statement of the problem

Green functions

Solutions of the **Wave Equation:**

$$(\nabla^2 + k^2)G(\overline{r}, \overline{r'}) = -\delta(|(\overline{r} - \overline{r'}|)$$

In **free space** they have the following **deterministic formulation:**

$$G_{f_{s2D}}(\vec{r}, \vec{r}') = -\frac{1}{4j}H_0^{(2)}(k|\vec{r} - \vec{r}'|)$$

Objective: find a stochastic development for the Green function.

Hypothesis and proof

- We suppose that the eigenvalues k_i of the Wave Equation are distributed following and uniform distribution, $k_i \sim U(0, K^2)$.

$$f_{k_i}(x) = \begin{cases} \frac{2}{K^2}x \; 0 \le x \le K \\ 0 \quad \text{en otro caso} \end{cases}$$

- We write G_{2D} in the following form:

$$G_{2D} = -\frac{1}{S}\sum Y_i^{(2D)}\Omega_i^2$$

where $\Omega_i \sim N(0, 1) \Rightarrow \Omega_i^2 \sim \chi_1^2$, and being:

$$Y_i^{(2D)} = \frac{J_0(k_i|\vec{r} - \vec{r}'|)}{k^2 - k_i^2}$$

Heavy-Tailed distribution

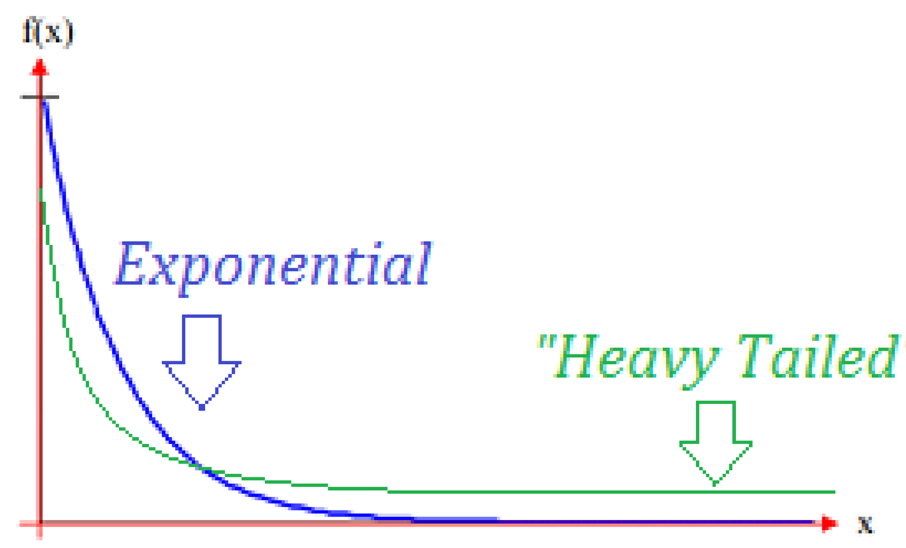
Definition

$$F_Y(y) = c|y|^{-\alpha} \quad \text{if } y \to -\infty$$

$$\overline{F}_Y(y) = 1 - F_Y(y) = c|y|^{-\alpha} \quad \text{if } y \to +\infty$$

Problems of f_Y

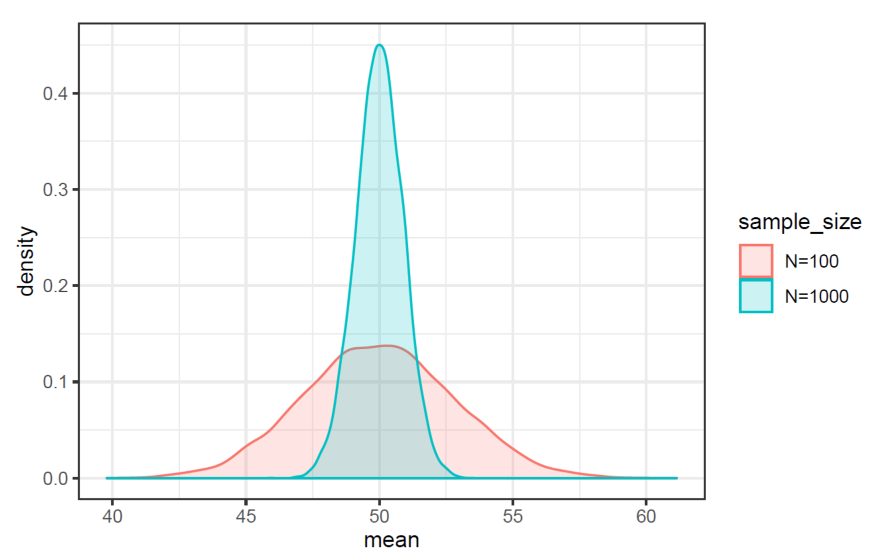
- The **mean** and the **variance** are not defined.
- It is not possible to apply either the **Law of Large Numbers** or the **Central Limit Theorem**.

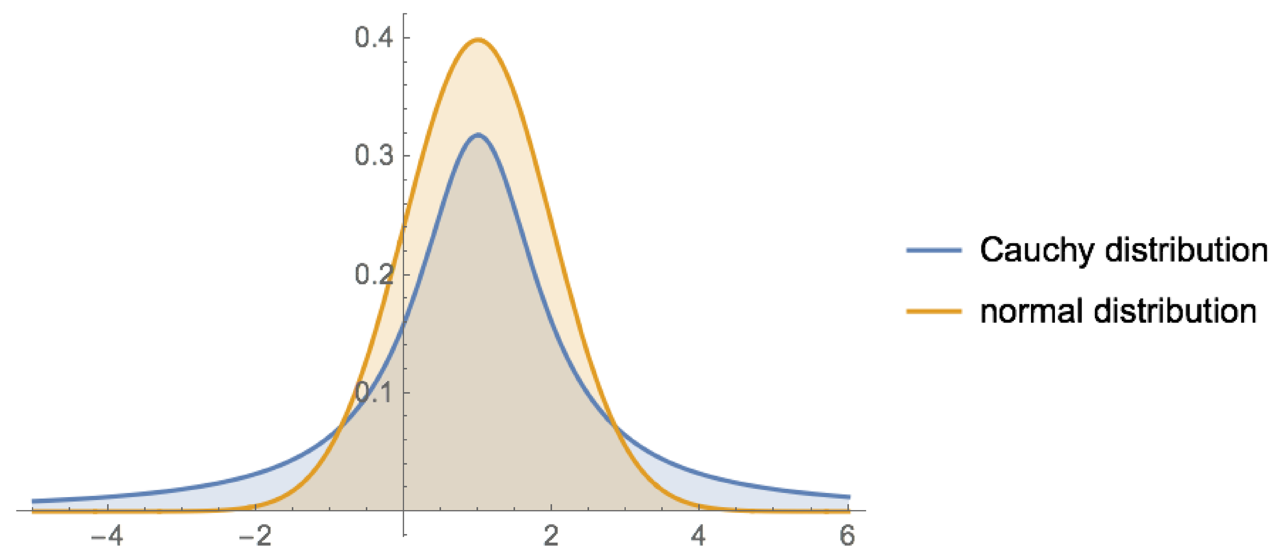


Generalization of the Central Limit Theorem

GCLT Gnedenko-Kolmogorov

The sum of independent and identically distributed variables, and symmetric, with tails tending to $|x|^{-\alpha-1}$, follows an stable distribution $S(t; \alpha, 0, x_0, \gamma)$.
If $\alpha > 2$, it is a **normal distribution**, and if $\alpha = 1$, it is a **Cauchy** distribution.





Results

Adjusted distribution: $G_{2D}(\vec{r}, \vec{r}') = G_{2D}(\vec{r}, \vec{r}') = -\frac{1}{S}\sum_i \frac{\omega_i^2}{k^2 - k_i^2}J_0(k_i|\vec{r} - \vec{r}'|) \sim C(x_0, \gamma)$,
with situation parameter x_0 and scale parameter γ:

$$x_0 = -\frac{1}{4}Y_0(k|\vec{r} - \vec{r}'|), \qquad \gamma = \left|\frac{J_0(k|\vec{r} - \vec{r}'|)}{4}\right|$$

Cauchy distribution

The Cauchy distribution $C(x_0, \gamma)$ is one whose density function is given by:

$$f(x) = \frac{\gamma}{\pi}\frac{1}{\gamma^2 + (x - x_0)^2}$$

1. It is a **symmetric** distribution.
2. Its **mean** μ and its **variance** σ^2 **are not defined**.
3. Its **moments** are not defined either.
4. Its **median** and its **mode** are equal, and the value is x_0.

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